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A corollary of the Poincaré–Bendixson theorem and periodic canards

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ABSTRACT

The role of topological methods in the analysis of canard-type periodic trajectories is discussed. A special corollary of the Poincaré–Bendixson theorem is used to prove the existence of periodic planar canards.

1. Main result

This paper investigates the role of topological methods in the analysis of canard-type periodic trajectories. We apply a special corollary of the Poincaré–Bendixson theorem [1,2] to the existence of periodic planar canards.

Consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a) \tag{1}$$

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with a smooth $\mathbf{f}$. Here $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and $a \in [a_-, a_+]$ is a parameter. Let Γ be a Jordan curve that bounds an open domain D. (A Jordan curve is a plane curve which is topologically equivalent to the unit circle, i.e., it is simple and closed.) Suppose that for any $a \in [a_-, a_+]$ there exists a unique equilibrium $\mathbf{e}_a \in D \cup \Gamma$, and

$$\det J(\mathbf{e}_a) > 0, \quad a_- \leqslant a \leqslant a_+, \tag{2}$$

where J denotes the Jacobian. We also suppose that

$$\operatorname{tr} J(\mathbf{e}_{a_-}) \cdot \operatorname{tr} J(\mathbf{e}_{a_+}) < 0, \tag{3}$$

where $\operatorname{tr} J$ stands for the trace of the Jacobian. Note that (2) and (3) imply that a Hopf bifurcation occurs for some value of parameter a between a_- and a_+.

Proposition 1.1. *Let system* (1) *have no cycles confined in $D \cup \Gamma$ for $a = a_-, a_+$. Then for some $a \in (a_-, a_+)$ there exists a cycle of system* (1) *which is confined in $D \cup \Gamma$, and which touches Γ.*

Of course, the gist of this statement is in the last three words: "*... which touches Γ*". We need to note that the cycle, which existence is guaranteed by Proposition 1.1, is not necessarily isolated. This proposition is a corollary of the Poincaré–Bendixson theorem, see the next section for a proof.

We present below four simple examples to demonstrate the role of Proposition 1.1 in analysis of periodic planar canards [3]. Poincaré–Bendixson theorem has already been used a number of times in proving the existence of planar canards. The most spectacular result is the one presented in [4]. However, our way to use the Poincaré–Bendixson theorem differs from those suggested previously, and our method is not immediately applicable to co-existence of several periodic orbits.

Example 1. Consider the system

$$\dot{x} = y, \qquad \varepsilon \dot{y} = -x + F(y + a) \tag{4}$$

with a small positive ε. Suppose that $F(0) = 0$, $F'(y) < 0$ for $y < 0$ and $F'(y) > 0$ for $y > 0$. The curve $x = F(y)$ is a slow curve of system (4) for $a = 0$. The branch $x = F(y)$, $y < 0$, is the attractive part of the slow curve, and the branch $x = F(y)$, $y > 0$, is the repulsive part. The origin is the turning point. Periodic canards are periodic solutions of system (4) which follow for substantial distance the repulsive branch, see Fig. 1. We say that at $a = 0$ system (4) has a family of periodic canards of magnitude $\alpha > 0$, if to any small $\varepsilon > 0$ one can correspond a_ε and a periodic solution $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ of the system $\dot{x} = y$, $\varepsilon \dot{y} = -x + F(y + a_\varepsilon)$, such that

$$\max\bigl\{x_{\varepsilon,a_\varepsilon}(t)\colon\ y_{\varepsilon,a_\varepsilon}(t) = 0\bigr\} = \alpha. \tag{5}$$

In our case a periodic solution may visit the upper half-plane $y > 0$ only traveling along the repulsive branch of the slow curve. Thus, this definition is consistent with the informal explanation given above. Below we sometimes omit the word "family" and talk about canards of magnitude α instead.

Proposition 1.2. *There exists a periodic canard of system* (4) *of any given magnitude $\alpha > 0$.*

Proof. Note that for any value of a the only equilibrium is given by

$$\mathbf{e}_a = \bigl(F(a), 0\bigr). \tag{6}$$

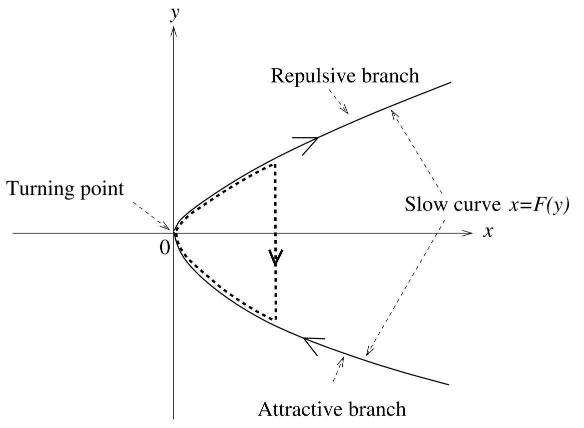

Fig. 1. Attractive and repulsive branches of the slow curve, and an example of a periodic canard (dotted line).

Thus

$$J(\mathbf{e}_a) = \begin{pmatrix} 0 & 1 \\ -1/\varepsilon & F'(a)/\varepsilon \end{pmatrix},$$

and the inequalities

$$\det J(\mathbf{e}_a) = 1/\varepsilon > 0, \qquad \operatorname{tr} J(\mathbf{e}_a) = F'(a)/\varepsilon < (>)\, 0 \quad \text{for } a < (>)\, 0 \tag{7}$$

follow.

The boundaries $a_- < 0 < a_+$ are chosen to be any numbers satisfying

$$-\alpha + F(a_-) < 0 \quad \text{and} \quad -\alpha + F(a_+) < 0. \tag{8}$$

Let us describe the domain D, see Fig. 2. We choose a number $\beta > \alpha$ such that the triangle

$$\Delta(\alpha, \beta) = \big\{(x, y)\colon\ \alpha \leqslant x \leqslant \beta,\ |y| \leqslant x - \alpha\big\}$$

belongs to the area where $-x + F(y + a) < 0$ for $a_- \leqslant a \leqslant a_+$. That is,

$$-x + F(y + a) < 0 \quad \text{for } (x, y) \in \Delta(\alpha, \beta),\ a \in [a_-, a_+]. \tag{9}$$

Existence of such β follows from (8). We also choose the numbers $\zeta_- < 0 < \zeta_+$ satisfying

$$-x + F(\zeta_\pm + a) > 0 \quad \text{for } x < \beta,\ a_- < a < a_+. \tag{10}$$

Consider the open quadrangle Q which is bounded from south and from north by the lines $y = \zeta_-$ and $y = \zeta_+$, bounded from east by the line $x = \beta$, and bounded from south-west by the line $x + y = \zeta_-$. Denote by D the open set $Q \setminus \Delta(\alpha, \beta)$, and denote by Γ the boundary of D.

Note that the equilibria $\mathbf{e}_a$, $a \in [a_-, a_+]$, belong to $D \cup \Gamma$ by (6); thus $a_- < 0 < a_+$ and (7) guarantee that (2) and (3) hold. To apply Proposition 1.1 it remains to show that there are no cycles confined in $D \cup \Gamma$ for $a \in \{a_-, a_+\}$.

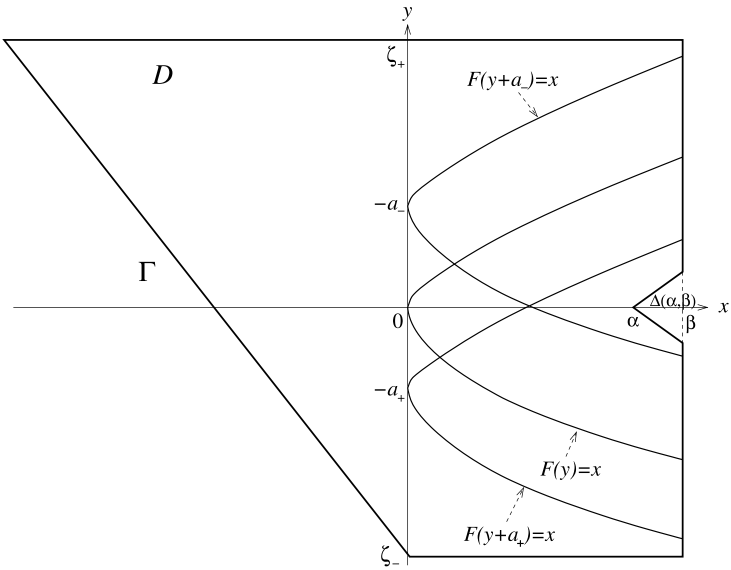

Fig. 2. The domain D is bounded by the curve Γ. The triangle $\Delta(\alpha,\beta)$ belongs to the area where $-x + F(y+a) < 0$ for all $a \in [a_-, a_+]$. The domain D contains all slow curves $x = F(y+a)$, $x \in [0,\beta]$, $a \in [a_-, a_+]$.

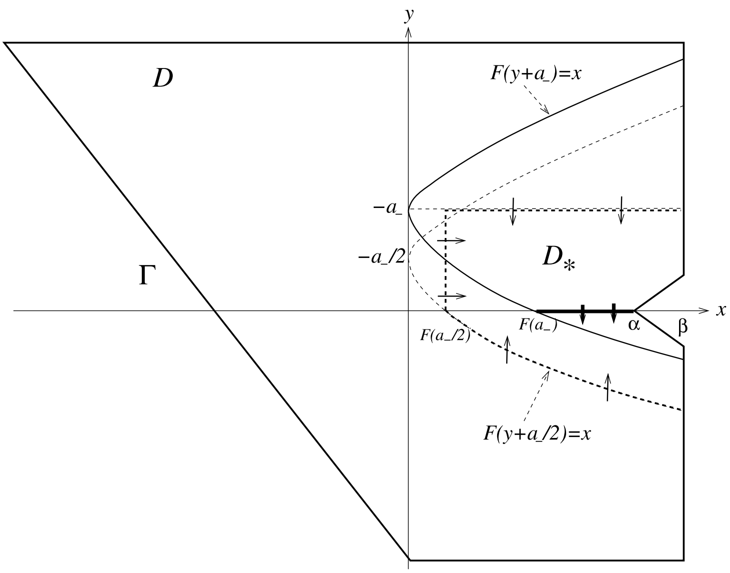

Fig. 3. The sub-domain $D_* \subset D$ is bounded by the bold dashed line. A trajectory which is confined in D cannot leave the domain D_*, once it entered D_*. Each periodic solution $\mathbf{x}_*(t) = (x_*(t), y_*(t))$ which is confined in D must visit D_*, because it must cross the bold segment of the axis $y = 0$. The areas are shrinking within D_*, and thus there are no cycles there.

Consider the case $a = a_-$. Introduce the auxiliary sub-domain $D_* \subset D$ which is bounded from north by the line $y + a_- = 0$, from west by the line $x = F(a_-/2)$ and from south-east by the graph of the function $-x + F(y + a_-/2) = 0$, see Fig. 3. *For small ε a trajectory $\mathbf{x}_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ which is confined in D cannot leave the domain D_*, once it entered D_*.* To prove this claim, we note that for the small ε the velocity vectors $\dot{\mathbf{x}}$ point inward at the part of the boundary of D_* which belongs

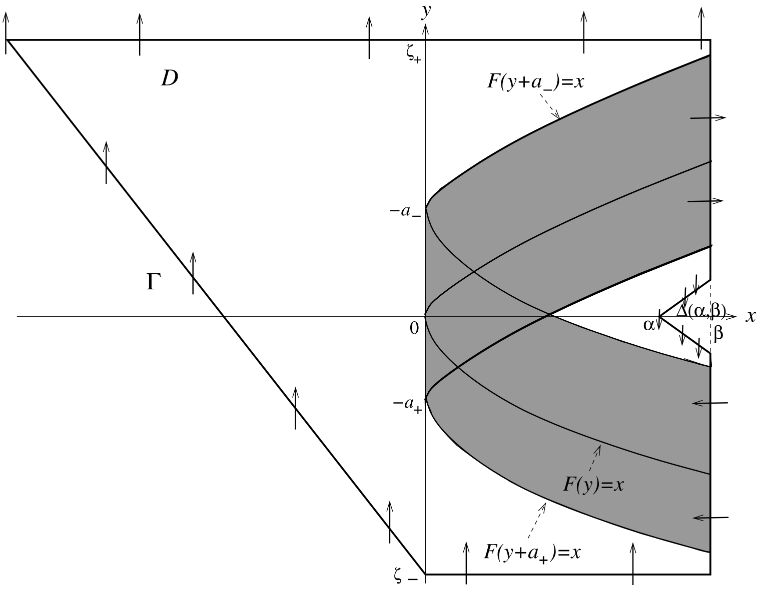

Fig. 4. A periodic orbit which is confined in $D \cup \Gamma$ may touch Γ only at the point $(\alpha, 0)$, because at all other points of Γ at least one end of the velocity vector points strictly outward Γ.

to D. Indeed, at the west boundary we have $\dot{x} = y > 0$; at the north boundary the inequality $\dot{y} = (-x + F(y + a_-))/\varepsilon < 0$ holds, and at the south-west boundary the velocity vectors point almost vertically up for small ε. *Moreover, each periodic solution* $\mathbf{x}_*(t) = (x_*(t), y_*(t))$ *which is confined in D must visit* D_*. Indeed, because $\dot{x} = y$, the solution $\mathbf{x}_*(t)$ must visit both half-plane $y < 0$ and half-plane $y > 0$. Thus $\mathbf{x}_*(t)$ must cross at some moments of time the axis $y = 0$ from above, i.e., for $x > x_{a_-} = F(a_-/2)$; it remains to note that the whole interval

$$\{(x, 0)\colon\ F(a_-/2) \leqslant x < \alpha\}$$

belongs to D_*.

By the italicized parts of the previous paragraph, each cycle which is confined in D must be confined in D_*. However, within D_* the inequality $\operatorname{tr} J(x, y) = F'(y + a_-)/\varepsilon < 0$ holds, the areas are shrinking, and therefore there are no cycles there. The case $a = a_-$ is completed, and the case $a = a_+$ can be considered analogously in the backward time.

Thus, by Proposition 1.1, for any small $\varepsilon > 0$ there exists a periodic solution $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ whose trajectory is confined in $D \cup \Gamma$ and touches Γ. On the other hand, a periodic orbit which is confined in $D \cup \Gamma$ may touch Γ only at the point $(\alpha, 0)$, see Fig. 4: at any other point at least one end of the velocity vector points strictly outward Γ. (At the north, south and south-west parts of the boundary this is due to almost upward orientation of $\dot{\mathbf{x}}$ for small ε, see (9); at the sides of the triangle $\Delta(\alpha, \beta)$, apart of the point $(0, \alpha)$ — due to almost downward orientation of $\dot{\mathbf{x}}$, see (10); and at the vertical fragments of the east boundary — due to $\dot{x} = y \neq 0$.) Thus, $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ is a family of periodic canards of the required magnitude α, and the proof is completed. $\square$

Statements similar to Proposition 1.2 provide no information about asymptotic of a_ε, and on stability of canards. Still they could be useful in applications: the canards which existence is known can be further localized and stabilized via a suitable feedback in a usual way. Also note that we do not guarantee that a canard of the magnitude α has only one interval of fast motion per period. The structure of a canard may be trickier, see Fig. 5.

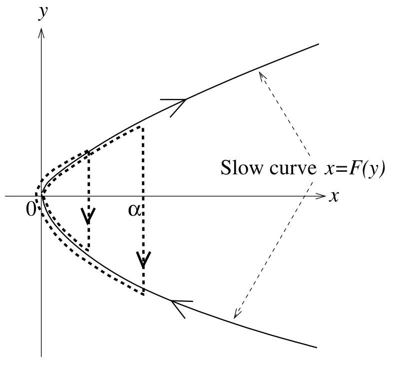

Fig. 5. Attractive and repulsive branches of the slow curve, and an example of a periodic canard (dotted line).

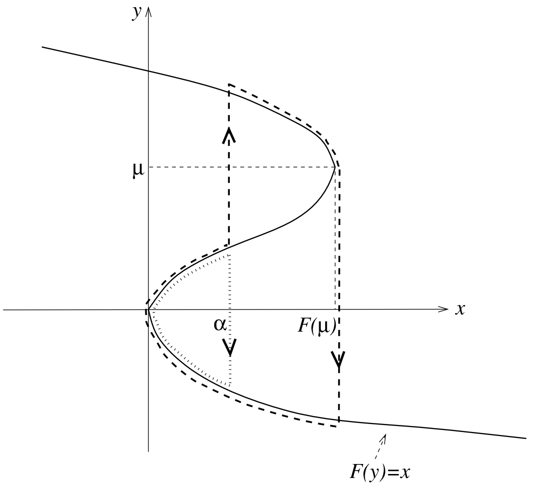

Fig. 6. Headless canard (dotted) and headed canard (dashed) for a bi-modal function $F(x)$.

Example 2. Consider system (4) with a bimodal function f. Suppose that $F(0) = 0$, $F'(y) < 0$ for $y < 0$ and for $y > \mu > 0$, whereas $F'(y) > 0$ for $0 < y < \mu$. The curve $x = F(y)$ is a slow curve of system (4) for $a = 0$. In particular, the branches $x = F(y)$, $y < 0$, and $x = F(y)$, $y > \mu$, are the attractive parts of the slow curve, and the branch $x = F(y)$, $0 < y < \mu$, is the repulsive part. The origin and the point $(F(\mu), \mu)$ are the turning points. This modification of the first example is similar to the classical Liénard equation.

Periodic canards of a magnitude α may exist only for $0 < \alpha \leqslant F(\mu)$. Moreover, there are two possible structures of a canard: solution may jump down, or jump up from the unstable part of the slow curve. We call such canards headless and headed, respectively. More formally, we say that system (4) has a headless periodic canard of magnitude $\alpha > 0$ at $a = 0$, if the relationship (5) holds. We say that the system has a headed periodic canard of magnitude $\alpha > 0$, if instead

$$\min\bigl\{x_{\varepsilon,a_\varepsilon}(t)\colon\ y_{\varepsilon,a_\varepsilon}(t) = \mu\bigr\} = \alpha. \tag{11}$$

Proposition 1.3. *There exist a headless and a headed periodic canard of any given magnitude $\alpha \in (0, F(\mu))$ (see Fig. 6).*

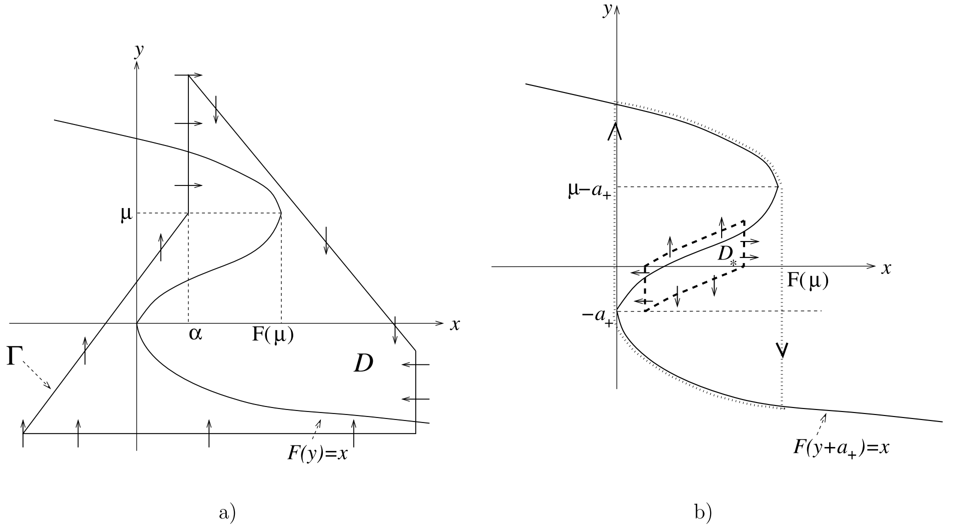

Fig. 7. a) Schematic image of a suitable region D bounded by the curve Γ. b) To prove non-existence of cycles at $a = a_+$ we consider region D_*. This region is a repeller, and it doesn't embrace any cycle, since the areas are growing within. Thus any cycle should cross the horizontal axis to the right of D_*, and for small ε must be close to the relaxation cycle (dotted). However for small a_+ this relaxation cycle does not belong to the region D.

Proof is similar to that of Proposition 1.2. As $a_\pm$ one can choose any small numbers satisfying $a_- < 0 < a_+$; the inequalities (2), (3) are evident. A possible construction of the region D is given in Fig. 7a). Non-existence of cycles at $a = a_-$ may be proven as before. For non-existence of cycles at $a = a_+$ see Fig. 7b).

Example 3. As our next example we consider system (4) where $F(y)$ is a smooth piecewise-monotone function which satisfies the following conditions

$$F(0) = 0, \qquad F(y) > 0, \quad \text{for } y \neq 0, \qquad \lim_{y\to\pm\infty} F(y) = \infty. \tag{12}$$

We also suppose that all local extrema of this function are pairwise different.

For a given $x_0 > F(y_0)$ we say that at $a = 0$ system (4) has a headless (x_0, y_0)-periodic canard, if to any small $\varepsilon > 0$ one can correspond a_ε and a periodic solution $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ of the system $\dot{x} = y$, $\varepsilon\dot{y} = -x + F(y + a_\varepsilon)$, such that

$$\max\{x_{\varepsilon,a_\varepsilon}(t)\colon\ y_{\varepsilon,a_\varepsilon}(t) = y_0\} = x_0. \tag{13}$$

Headed canards for the case $x_0 < F(y_0)$ are defined in the same way, with the difference that (13) is replaced by

$$\min\{x_{\varepsilon,a_\varepsilon}(t)\colon\ y_{\varepsilon,a_\varepsilon}(t) = y_0\} = x_0. \tag{14}$$

Introduce an auxiliary function F^*, see Fig. 8b),

$$F^* = \begin{cases} \min_{z\geqslant y} F(z), & y \geqslant 0, \\ \min_{z\leqslant y} F(z), & y < 0. \end{cases}$$

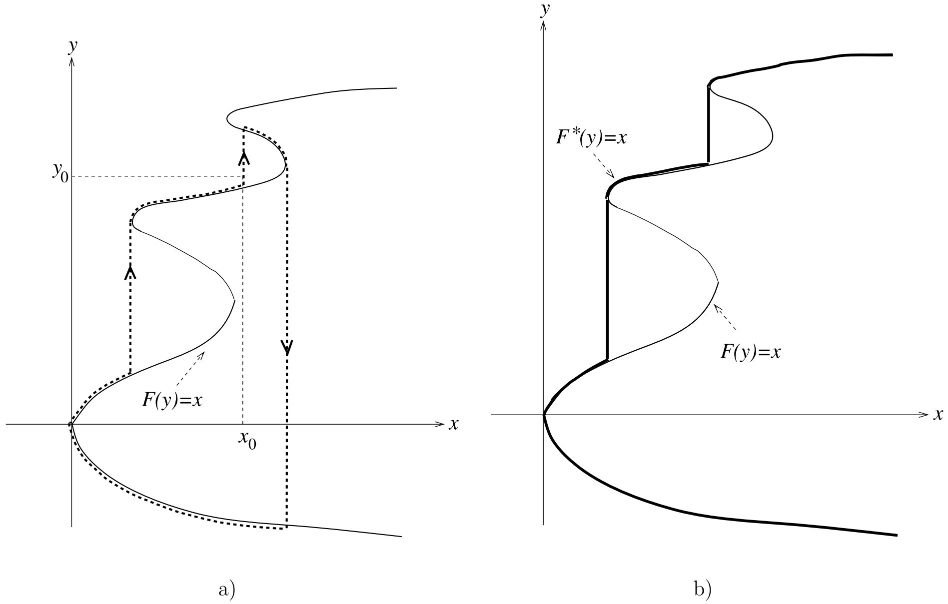

Fig. 8. a) An example of a "multi-headed" canard for a multi-mode function $F(y)$. b) The line $x = F^*(y)$ (bold) versus the line $x = F(y)$. Headless (x_0, y_0)-canards exist for any point (x_0, y_0) located to the right of the line $x = F(y)$; headed (x_0, y_0)-canards exist for any point (x_0, y_0) located strictly between the lines $x = F(y)$ and $x = F^*(y)$.

Proposition 1.4. *There exists a headless (x_0, y_0)-periodic canard for any $x_0 > F(y_0)$, and there exists a headed (x_0, y_0)-periodic canard for any $F(y_0) < x_0 < F^*(y_0)$.*

The structure of such periodic canards is illustrated in Fig. 8a). The proof combines the proofs of two previous propositions.

Example 4. As the last example we consider the system

$$\dot{x} = F(x, y) = x\big(p - f(y)\big), \qquad \varepsilon\dot{y} = G(x, y, a) = y\big(-q + x\big(r + g(y) - ah(y)\big)\big). \tag{15}$$

Here $p, q, r > 0$ are given numbers, $f(0) = g(0) = h(0) = 0$, $f'(y), g'(y), h'(y) > 0$ for $y \geqslant 0$, ε is small, and a is a parameter. Suppose also that

$$\lim_{y\to\infty} g(y)/h(y) = 0. \tag{16}$$

System (15) has been recently used in population dynamics. Loosely speaking, the functions $r + g(y)$ and $h(y)$ describe facilitation and competition between predators, respectively. Eq. (16) means that the competition prevails for denser populations of predators. The small parameter ε manifests that the metabolism rate of the prey is significantly slower than that of the predator. This is the case, e.g., for interactions between bacteria and fages, see [7].

An instructive example of the functions $g(y), h(y)$ is given by

$$g(y) = \alpha_1 y + \alpha_2 y^2 + \cdots + \alpha_m y^m, \qquad h(y) = \beta_1 y^{m+1} + \beta_2 y^{m+2} + \cdots + \beta_n y^{m+n}, \tag{17}$$

where all coefficients are non-negative, and at least on α_i and at least one β_j is strictly positive. Loosely speaking, α_i measure intensity of mutual facilitation between $i+1$ predators, whereas β_j measure intensity of competition between $m+i+1$ predators. Another similar example is given by

$$g(y) = \int_0^M v(\alpha)\,d\alpha, \qquad h(y) = \int_M^N w(\alpha)\,d\alpha, \tag{18}$$

where the weight functions $v(\alpha), w(\alpha)$ are positive and bounded, and $0 < M < N$.

The system of equation to find "canard-susceptible" triplets (x_*, y_*, a_*) is

$$F(x, y, a) = 0, \qquad G(x, y, a) = 0, \qquad G'_y(x, y, a) = 0.$$

In the positive quadrant $x, y > 0$ this can be rewritten as

$$f(y) = a, \qquad x\bigl(r + g(y) - ah(y)\bigr) = q, \qquad g'(y) = ah'(y).$$

Since $f(0) = 0$ and $f'(y) > 0$ for $y \geqslant 0$, there exists a unique $y_* > 0$ which satisfies $f(y) = a$; thus $a_* = g'(y_*)/h'(y_*)$, and $x_* = q/(r + g(y_*) - a_*h(y_*))$. We suppose that x_* is positive, that is that the inequality

$$r + g(y_*) > a_*h(y_*)$$

holds.

In the positive quadrant the slow curve is given by

$$x = X(y) = q/\bigl(r + g(y) - a_*h(y)\bigr), \quad 0 < y < \eta. \tag{19}$$

To avoid non-principal complications we suppose that the function $g'(y)/h'(y)$, strictly decreases for $y > 0$; this is always true in the case (17) or (18). (For instance, in the case (17) we rewrite $g'(y)/h'(y)$ as $g_1(y)/h_1(y)$ with $g_1(y) = g'(y)/y^m, h_1(y) = h'(y)/y^m$; then $g_1(y)$ strictly decreases, $h_1(y)$ strictly increases, and the fraction $g'(y)/h'(y) = g_1(y)/h_1(y)$ strictly decreases as required.) Then, in particular, the function $r + g(y) - a_*h(y)$ is unimodal, and there exists the single positive root η of the equation $r + g(y) - a_*h(y) = 0$. Therefore the function (19) is unimodal for $0 < y < \eta$. The branch $x = X(y)$, $0 < y < y_*$, is repulsive, the branch $x = X$, $y_* < y < \eta$, is attractive, and (x_*, y_*) is the unique turning point.

For $a \sim a_*$ the system may have two types of canards, see Fig. 9. Headless canards may exist for $x_0, y_0 > 0$ satisfying $X(y_0) < x_0 < X(0)$, and they have the standard structure. Headed canards, which may exist for $x_0, y_0 > 0$ satisfying $0 < y_0 < y_*$, $x_* < x_0 < X(y_0)$, are more interesting. A headed (x_0, y_0)-canard exhibits additional delayed loss of stability phenomenon: after following down at $x \approx x_0$ it follows closely the axis $y = 0$ until a point $x \approx \xi_0 > q/r$, and then jumps up to the attractive branch of the slow manifold. The point ξ_0 is the solution of the equation $\xi^q \exp(-r\xi) = x_0^q \exp(-rx_0)$. Indeed, close to the axis $y = 0$ the dynamics is governed by the equation $dy/dx = y(q - rx)/(px)$ whose solutions satisfy the relationship $\ln y^p - \ln x^q + rx = \text{const}$.

Proposition 1.5. *There exists a headless (x_0, y_0)-periodic canard for any $x_0, y_0 > 0$ satisfying $X(y_0) < x_0 < X(0)$, and there exists a headed (x_0, y_0)-periodic canard for any $x_0, y_0 > 0$ satisfying $0 < y_0 < y_*, x_* < x_0 < X(y_0)$.*

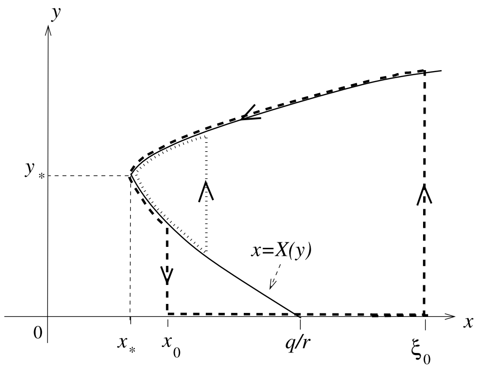

Fig. 9. Headless and headed canards for the modified Lotka–Volterra system. Headless canards may exist for $x_0, y_0 > 0$ satisfying $X(y_0) < x_0 < X(0)$, and they have the standard structure. Headed canards, which may exist for $x_0, y_0 > 0$ satisfying $0 < y_0 < y_*$, $x_* < x_0 < X(y_0)$, are more interesting. A headed (x_0, y_0)-canard exhibits additional delayed loss of stability phenomenon: after following down at $x \approx x_0$ it follows closely the axis $y = 0$ until the point ξ_0 and then jumps up to the attractive branch of the slow manifold.

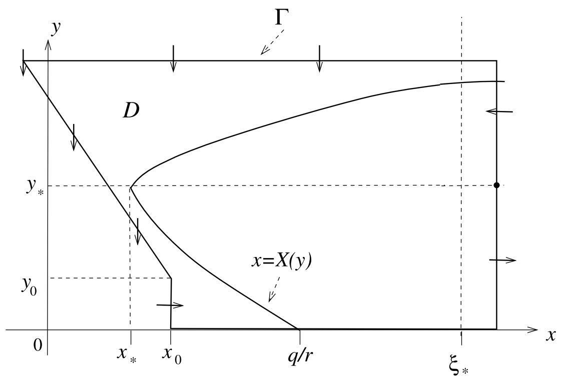

Fig. 10. The region D bounded by the curve Γ. At last one endpoint of the velocity vector points strictly outward of D at all points of Γ, except of two points: one is our "target point" (x_0, y_0), and another is the bold point at the eastern bound of D. However, there are no cycles which touch Γ at the second point, since the longest possible travel along the axis $y = 0$ ends at the point $2q/r - x_*$, which is located strictly to left of the eastern bound of D. Thus, by Proposition 1.1, there exists a cycle which is confined in $D \cup \Gamma$, and touches Γ at (x_0, y_0).

Proof. As $a_\pm$ we choose any numbers which are sufficiently close to a_* and satisfy $a_- < a_* < a_+$. Note that

$$J(\mathbf{e}_a) = \begin{pmatrix} 0 & -x_a f'(y_*) \\ y_*(g(y_*) - ah(y_*))/\varepsilon & x_a y_*(g'(y_*) - ah'(y_*))/\varepsilon \end{pmatrix}$$

and the inequalities (2) and (3) follow. The construction of the region D in the case of a headless canard is the same as in the first example, and in the case of a headed canard is explained in Fig. 10. Here ξ_* denotes the unique positive solution of the equation $\xi^q \exp(-r\xi) = x_*^q \exp(-rx_*)$. Nonexistence of confined in $D \cup \Gamma$ cycles for $a = a_-, a_+$ can be proven as in the previous examples. $\square$

The results of this paper are relevant to the use of topological degree in analysis of canards in multi-dimensional systems [5]; they were formulated in our preprint [6].

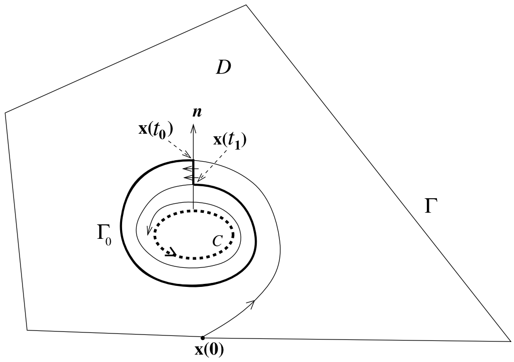

Fig. 11. The trapping curve Γ_0.

2. Proof of Proposition 1.1

The Poincaré–Bendixson theorem can be stated in several ways. The statement that is relevant to Eq. (1) is the following. *Suppose S is a closed, bounded subset of the plane*; *S does not contain any fixed points*; *and there exists a trajectory confined in S. Then either this trajectory is a closed orbit, or it spirals toward a closed orbit.*

For a particular value of a a solution $\mathbf{x}(t)$ of (1) is called directed, if $\mathbf{x}(0) \in \Gamma$ and $\mathbf{x}(t) \in \bar{D}$ for $t > 0$. *There exists a directed solution* $\mathbf{x}(t)$ *for* $a = a_+, a_-$. To prove this claim we suppose that $\operatorname{tr} J(\mathbf{e}_{a_-}) < 0$, and consider a solution which begins in a sufficiently small vicinity of $\mathbf{e}_{a_-}$. If $y(0) = \mathbf{e}_{a_-}$, we have nothing to prove. Otherwise, $|\mathbf{y}(t) - \mathbf{e}_{a_-}| \to 0$ as $t \to \infty$, and $|\mathbf{y}(t)|$ is bounded from below by a strictly positive constant at $t \leqslant 0$ (because $\mathbf{e}_{a_-}$ is a sink due to $\det J(\mathbf{e}_{a_-}), \operatorname{tr} J(\mathbf{e}_{a_-}) < 0$). By the Poincaré–Bendixson theorem $\mathbf{y}(t)$ must leave D in negative time (because there is no cycles at $a = a_-$); in particular, $\mathbf{y}(t)$ touches Γ for the first time at some $t = \tau < 0$. It remains to set $\mathbf{x}(t) = \mathbf{y}(t + \tau)$. Analogously, using the backward time, we prove that *there are no* Γ*-directed solutions at* $a = a_+$.

Denote by $a_0 \in [a_-, a_+)$ the lowest upper bound of $a \in [a_-, a_+)$ for which there exist some directed solutions. For $a = a_0$ a directed solution $\mathbf{x}_0(t)$ also exists by continuity. If $\mathbf{x}_0(\cdot)$ is periodic, then the proposition holds. To finalize the proof *we suppose that* $\mathbf{x}(\cdot)$ *is not periodic, and arrive at contradiction.*

By the Poincaré–Bendixson theorem there are only two possibilities: either

(a) $\mathbf{x}_0(t)$ spirales toward C a cycle $C \subset D$,

or

(b) $\mathbf{x}_0(t) \to \mathbf{e}_{a_0}$ for $t \to \infty$.

Let $\Gamma_0 \subset D$ be a Jordan curve which bounds the open domain D_0, and τ be a positive number. We say that the pair $\{\Gamma_0, \tau\}$ is *trapping* if simultaneously: the set $D_0 \cup \Gamma_0$ is forward invariant for the equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a_0)$, and $\mathbf{x}(\tau) \in D_0$ holds for any solution satisfying $\mathbf{x}(0) \in \Gamma_0$. If a trapping pair exists, then for a slightly greater than a_0 the solutions of Eq. (1) that begins at $\mathbf{x}_0(0)$ are also attracted to arbitrary small vicinity of D_0. That is, there exist a directed solutions at some $a > a_0$. Thus, to arrive at contradiction it is enough to construct a trapping pair. To this end in the case (a) we choose a point $\mathbf{y} \in C$ and consider the corresponding outward normal $\mathbf{n}$ to C. Let $\bar{\lambda}$ satisfies the relationships $[\mathbf{y}, \mathbf{y} + \bar{\lambda}\mathbf{n}] \subset D$, and $\mathbf{f}(\mathbf{y} + \lambda\mathbf{n}, a_0) \cdot \mathbf{f}(\mathbf{y}, a_0) > 0$, $0 \leqslant \lambda \leqslant \bar{\lambda}$. By definition, the solution $\mathbf{x}_0(t)$ crosses the segment $(\mathbf{y}, \mathbf{y} + \bar{\lambda}\mathbf{n}]$ infinitely many times, see Fig. 11. Let t_0 and t_1 be two successive moments of such crossings with the corresponding values λ_0, λ_2. Consider the curve Γ_0 which consists of the trajectory $\mathbf{x}_0(t)$, $t_0 < t < t_1$, together with the segment $[\mathbf{x}_0(t_0), \mathbf{x}_0(t_1)]$. Since $\mathbf{x}_0(t)$ spirales toward C, the inequality $\lambda_0 > \lambda_1$ holds. Therefore, $\{\Gamma_0, t_1 - t_0 + 1\}$ is a trapping pair, and we arrived at contradiction in the case (a).

By $\operatorname{tr} J(\mathbf{e}_{a_0}) < 0$ the case (b) can be partitioned in turn into three cases:

(b1) $\mathbf{e}_{a_0}$ is a source;
(b2) $\mathbf{e}_{a_0}$ is a sink;
(b3) $\mathbf{e}_{a_0}$ is a center in the linear approximation.

In the case (b1) we immediately arrive at contradiction with the condition (b). In the case (b2), for a slightly greater than a_0, the solution which begins at $\mathbf{e}_{a_0}$ is attracted to a small vicinity of $\mathbf{e}_{a_0}$. Thus, there exist directed solutions for some $a > a_0$, which contradicts the definition of a_0. It remains to consider the case (b3), which is similar to the case (a) above. Indeed, consider a segment $\sigma = (\mathbf{e}_{a_0}, \mathbf{e}_{a_0} + \mathbf{z}]$ where $\mathbf{z}$ is close enough to $\mathbf{e}_{a_0}$ to guarantee that $\sigma \subset D$, and that $\mathbf{f}(\mathbf{y}, a_0)$, $\mathbf{y} \in \sigma$, is not collinear to $\mathbf{z}$. (This can be done because $\mathbf{e}_{a_0}$ is a center in the linear approximation.) By the condition (b) the solution $\mathbf{x}_0(t)$ crosses the segment σ infinitely many times. Let t_0 and t_1 be to successive moments of such crossings. Consider the curve Γ_0 which consists of the trajectory $\mathbf{x}_0(t)$, $t_0 < t < t_1$, together with the segment $[\mathbf{x}_0(t_0), \mathbf{x}_0(t_1)]$. By construction the pair $\{\Gamma_0, t_1 - t_0 + 1\}$ is a trapping pair, and we arrived at contradiction in the case (b3). The proposition is proven.

3. Non-smooth perturbations

Consider a perturbed system (4):

$$\dot{x} = y, \qquad \varepsilon \dot{y} = -x + F(y - a) + \tilde{F}(x, y, a),$$

where $\tilde{F}$ is continuous and small in the uniform norm: $\sup |\tilde{F}(x, y, a)| < \delta \ll 1$, but there are no bounds for its derivatives. In this case applicability of usual tools is doubtful.

Proposition 3.1. *There exist $\bar{\varepsilon}, \bar{\delta} > 0$ such that for $0 < \varepsilon < \bar{\varepsilon}$, $0 < \delta < \bar{\delta}$ there exist a small a_ε and a periodic* canard$(x_\varepsilon(t), y_\varepsilon(t))$ *of the system $\dot{x} = y$, $\varepsilon \dot{y} = -x + F(y - a_\varepsilon) + \tilde{F}(x, y, a_\varepsilon)$ which satisfies $\max_t\{x_\varepsilon(t)\} = b$. The trajectory of this canard approaches $\Gamma(b)$ as $\varepsilon, \delta \to 0$.*

Proof follows from the following modification of Proposition 1.1. Consider the equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a) + \tilde{\mathbf{f}}(\mathbf{x}, a). \tag{20}$$

Here $\tilde{\mathbf{f}}(\mathbf{x}, a)$ is continuous and uniformly small: $\sup \tilde{\mathbf{f}}(\mathbf{x}, a) < \delta \ll 1$, but there is no restriction on its derivative. *Under conditions of Proposition* 1.1 *for some $a \in (a_-, a_+)$ there exists a cycle of system* (20) *which belongs to $D \cup \Gamma$, and which touches Γ.* Proof is essentially the same as of Proposition 1.1.

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